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Math 210D
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Proof Portfolio

Problem 1

Let x and y be positive real numbers. If $x \neq y$, then $x + y > \frac{4xy}{x+y}$.

Proof We will assume that x and y are positive real numbers. We will proceed with a proof by contraposition. Therefore we will also assume that $x + y \leq \frac{4xy}{x+y}$ and will show that $x = y$.

Because both x and y are positive numbers, their sum will also be a positive number. Therefore we can safely multiply both sides of our inequality by $x + y$ to obtain

$$\begin{aligned} x + y &\leq \frac{4xy}{x+y} \\ (x+y)(x+y) &\leq \frac{4xy}{x+y}(x+y) \\ (x+y)(x+y) &\leq 4xy. \end{aligned}$$

Next we can expand the right side of our equation to get

$$x^2 + 2xy + y^2 \leq 4xy.$$

Now we can gather all of our terms on the right side by subtracting $4xy$ from both sides

$$\begin{aligned} x^2 + 2xy + y^2 - 4xy &\leq 4xy - 4xy \\ x^2 - 2xy + y^2 &\leq 0 \end{aligned}$$

which leaves us with 0 on the left side. Note that we can factor the right side to get

$$(x-y)^2 \leq 0.$$

We can solve for $x + y$ by taking the square root of both sides. Because squaring a quantity can turn negative values into positive ones, we must be careful with our inequality. Thus we are left with the following equations

$$x - y \leq 0 \text{ and } x - y \geq 0.$$

To find the relationship between x and y we can add y to both sides of the equations to get

$$\begin{aligned} x - y + y \leq 0 + y &\quad \text{and} \quad x - y + y \geq 0 + y \\ x \leq y &\quad \text{and} \quad x \geq y. \end{aligned}$$

The only way for $x \leq y$ and $x \geq y$ to be true is for $x = y$, as desired.

Thus we have proven that for all real numbers x and y, when $x + y \leq \frac{4xy}{x+y}$, then $x = y$; or equivalently when $x \neq y$, then $x + y > \frac{4xy}{x+y}$. ■

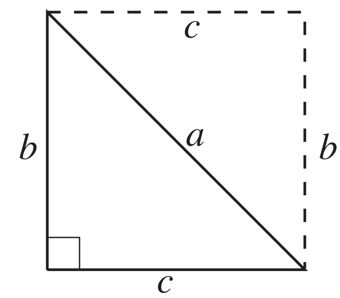

Figure 1: $\triangle ABC$ is a right, isosceles triangle

Problem 2

For all right triangles with a hypotenuse of length a cm and legs of length b cm and c cm, the triangle is isosceles if and only if its area is $\frac{1}{4}a^2$ cm^2.

Proof. We will assume $\triangle ABC$ is a right triangle with a hypotenuse of length a cm and legs of length b cm and c cm. We will show that $\triangle ABC$ is isosceles if and only if its area is $\frac{1}{4}a^2$ cm^2. To do this we will show two things. First, if $\triangle ABC$ is isosceles, then its area is $\frac{1}{4}a^2$ cm^2. Secondly, if $\triangle ABC$ has an area of $\frac{1}{4}a^2$ cm^2, then $\triangle ABC$ is isosceles.

We will start with proving that if $\triangle ABC$ is isosceles, then its area is $\frac{1}{4}a^2$ cm^2. To do this we will additionally assume that $\triangle ABC$ is isosceles and will show its area to be $\frac{1}{4}a^2$ cm^2. We can see a general picture of what $\triangle ABC$ might look like in Figure 1.

Note that because $\triangle ABC$ is a right triangle, can can flip it onto itself to make a rectangle. (See the dashed lines in Figure 1.) The area of this quadrilateral can be found by multiplying b by c. The area of $\triangle ABC$ would be have of that, or simply $\frac{1}{2}bc$.

The definition of isosceles tells us that the legs of our triangle will both have the same length, therefore $b = c$. We can use this fact to make a substitution in our area formula to obtain

$$\begin{aligned} \text{area of } \triangle ABC &= \frac{1}{2}bc \\ &= \frac{1}{2}b(b) \\ &= \frac{1}{2}b^2. \end{aligned}$$

Because $\triangle ABC$ is a right triangle, we can use the Pythagorean theorem which states $a^2 = b^2 + c^2$. Again, we know that $b = c$ so we can make another substitution to get

$$\begin{aligned} a^2 &= b^2 + c^2 \\ a^2 &= b^2 + b^2 \\ a^2 &= 2b^2. \end{aligned}$$

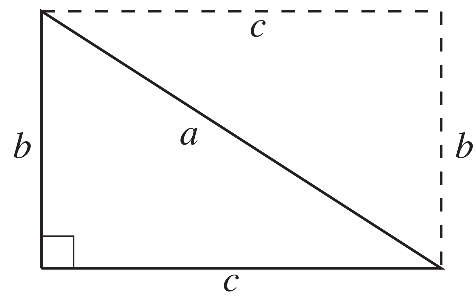

Figure 2: $\triangle ABC$ is a right triangle

Then we can divide by 2 to solve for b^2 which yields

$$\frac{1}{2}a^2 = b^2.$$

We can plug this value of b^2 into our area formula to get

$$\begin{aligned}\text{area of } \triangle ABC &= \frac{1}{2}b^2 \\ &= \frac{1}{2}(\frac{1}{2}a^2) \\ &= \frac{1}{4}a^2\end{aligned}$$

as desired.

Finally, we must show that if $\triangle ABC$ has an area of $\frac{1}{4}a^2$ cm^2, then $\triangle ABC$ is isosceles. Like before, we can rotate the triangle onto itself to make a rectangle (see Figure 2). The area of the rectangle would be half of this rectangle, thus the area of $\triangle ABC$ is also $\frac{1}{2}bc$. Now we know that

$$\frac{1}{4}a^2 = \frac{1}{2}bc.$$

We are still dealing with a right triangle, so by the Pythagorean Theorem we know that $a^2 = b^2+c^2$. Let's substitute this value for a^2 into our area equation to obtain

$$\begin{aligned}\frac{1}{4}a^2 &= \frac{1}{2}bc \\ \frac{1}{4}(b^2+c^2) &= \frac{1}{2}bc.\end{aligned}$$

We can simplify this equation by multiplying both sides by 4 and then grouping all terms on the same side to get

$$\begin{aligned}4 \cdot \frac{1}{4}(b^2+c^2) &= 4 \cdot \frac{1}{2}bc \\ b^2+c^2 &= 2bc \\ b^2+c^2-2bc &= 2bc-2bc \\ b^2-2bc+c^2 &= 0.\end{aligned}$$

Note that we can now factor the left side to obtain

$$(b-c)^2 \quad = \quad 0.$$

To solve for $b-c$ we will take the square root of both side which gives us

$$\begin{aligned} \sqrt{(b-c)^2} &= \sqrt{0} \\ b-c &= 0 \end{aligned}$$

or equivalently,

$$b=c.$$

If b and c are equal to each other, that means the legs of our triangle have the same length. Thus, by the definition of isosceles, $\triangle ABC$ is isosceles, as desired.

We have show that, for a right triangle $\triangle ABC$, both, if $\triangle ABC$ is isosceles then its area is $\frac{1}{4}a^2$ cm^2; as well as, if the area of $\triangle ABC$ is $\frac{1}{4}a^2$ cm^2, then $\triangle ABC$ is isosceles. Thereby we have proven that a right triangle $\triangle ABC$, with a hypotenuse of length a cm and legs of length b cm and c cm, is isosceles if and only if its area is $\frac{1}{4}a^2$ square centimeters. $\blacksquare$

Problem 3

If m is an odd integer, then the equation $x^2 + x - m = 0$ has no integer solution for x.

***Proof*.** We will prove this statement with an indirect proof by contraposition. Therefore we will assume that $x^2 + x - m = 0$ has an integer solution for x and will show that m is an even integer.

Assuming that $x^2 + x - m = 0$ has an integer solution for x means that

$$x^2 + x - m = 0$$

for some integer x. Let's manipulate our equation by adding m to both sides and doing a bit of factoring to get

$$\begin{aligned} x^2 + x - m &= 0 \\ x^2 + x &= m \\ x(x+1) &= m. \end{aligned}$$

Now we have two cases to consider. Because $x \in \mathbb{Z}$, x is either even or odd.

<u>Case 1</u>: First let's consider where x is even. Since 1 is odd, $x + 1$ would be odd as proven by Theorem 2 which states that the sum of an even and odd integer is an odd integer. In this case, the product of $x(x + 1)$ would be even because, as Theorem 4 states, the product of an even and odd integer is even. Thus m is even as desired.

<u>Case 2</u>: Next we must consider where x is odd. Since 1 is odd, $x + 1$ would be even as proven by Theorem 3 which states that the sum of any two odd integers is even. In this case again, the product of $x(x + 1)$ would be even because, as Theorem 4 states, the product of an even and odd integer is even. Thus m is even as desired.

In both cases we have shown that m is even. Therefore we have proven that if $x^2 + x - m = 0$ has an integer solution for x, then m is an even integer; or equivalently, if m is an odd integer, then $x^2 + x - m = 0$ has no integer solution for x. ■

Problem 4

For all natural numbers n, $5^n - 4n \equiv 1 \pmod{16}$.

Proof. We will assume that n is a natural number. We will proceed with a proof by mathematical induction. We will define our predicate as

$$P(n) : 5^n - 4n \equiv 1 \pmod{16}.$$

We will show $P(1)$ is true and that if $P(k)$ is true, then $P(k+1)$ is true for all natural numbers k.

First let's start by showing $P(1)$ is true. To do this, we must show that $5^1 - 4(1) \equiv 1 \pmod{16}$. Let's start by evaluating the left side of that equation:

$$\begin{aligned} 5^1 - 4(1) &= 5 - 4 \\ &= 1. \end{aligned}$$

We already have 1 in the right side. We know that congruence modulo 16 is an equivalence relationship and thus it is reflexive; therefore $1 \equiv 1 \pmod{16}$ is true. Consequently, $P(1)$ is true, as desired.

Next we must show that if $P(k)$ is true, then $P(k+1)$ is also true. Therefore we will assume that $P(k)$ is true and will show that $P(k+1)$ is true for any arbitrary $k \in \mathbb{N}$. We start by choosing a $k \in \mathbb{N}$. To show that $P(k+1)$ is true, we will show that $5^{k+1} - 4(k+1) \equiv 1 \pmod{16}$. We can write $P(k)$ as

$$5^k - 4k \equiv 1 \pmod{16}.$$

By the definition of congruence modulo 16, this means that

$$16 \mid 5^k - 4k - 1.$$

The definition of divides allows us to rewrite this statement as

$$16m = 5^n - 4n - 1$$

for some integer m. Let's start working toward $P(k+1)$ by multiplying both sides by 5. When we do this we get

$$\begin{aligned} 5 \cdot 16m &= 5(5^k - 4k - 1) \\ 80m &= 5 \cdot 5^k - 5 \cdot 4k - 5 \cdot 1 \\ 80m &= 5^{k+1} - 20k - 5. \end{aligned}$$

Now let's manipulate the right side a bit to obtain

$$80m = 5^{k+1} - 4k - 4 - 1 - 16k.$$

Further manipulation gives us

$$\begin{aligned} 80m &= 5^{k+1} - 4(k+1) - 1 - 16k \\ 80m + 16k &= 5^{k+1} - 4(k+1) - 1 \\ 16(5m+k) &= 5^{k+1} - 4(k+1) - 1. \end{aligned}$$

Note that m is an integer, so $5m$ is an integer as well because 5 is an integer and the integers are closed under multiplication. We also know that k is an integer, and, because the integers are closed under addition, $5m + k$ is an integer. The definition of divides allows us to rewrite the following statement as

$$16 \mid 5^{k+1} - 4(k + 1) - 1$$

which, by the definition of congruence modulo 16, also means

$$5^{k+1} - 4(k + 1) \equiv 1 \pmod{16}.$$

This means that $P(k + 1)$ is true, as desired.

Thus, by the First Principle of Mathematical Induction, we have proven that $5^n - 4n \equiv 1 \pmod{16}$ for all natural numbers n. $\blacksquare$

Problem 5

For all nonempty sets A, B, and C from some universal set U, $(A \times B) \cup (C \times B) = (A \cup C) \times B$.

Proof. We will assume that $A, B, C \subseteq U$. We will show that $(A \times B) \cup (C \times B) = (A \cup C) \times B$ for all nonempty sets A, B, C. To do this we will show that $(A \times B) \cup (C \times B) \subseteq (A \cup C) \times B$ and $(A \cup C) \times B \subseteq (A \times B) \cup (C \times B)$.

We will start by showing $(A \times B) \cup (C \times B) \subseteq (A \cup C) \times B$. Let's let choose an arbitrary $(x_1, y_1) \in (A \times B) \cup (C \times B)$. This means that $(x_1, y_1) \in (A \times B)$ or $(x_1, y_1) \in (C \times B)$.

Case 1: First let's consider $(x_1, y_1) \in (A \times B)$. This means that $x_1 \in A$ and $y_1 \in B$. Since $x_1 \in A$, then $x_1 \in A \cup C$. Now that we know $x_1 \in A \cup C$ and $y_1 \in B$, we can say $(x_1, y_1) \in (A \cup C) \times B$.

Case 2: In the other case, we must consider where $(x_1, y_1) \in (C \times B)$. This means that $x_1 \in C$ and $y_1 \in B$. Since $x_1 \in C$, then $x_1 \in A \cup C$. Thus we can say $(x_1, y_2) \in (A \cup C) \times B$ because $x_1 \in A \cup C$ and $y_1 \in B$.

In both cases we have shown that any element in $(A \times B) \cup (C \times B)$ is also in $(A \cup C) \times B$, therefore $(A \times B) \cup (C \times B) \subseteq (A \cup C) \times B$ as desired.

Now for the second part. We must show $(A \cup C) \times B \subseteq (A \times B) \cup (C \times B)$. Let's choose an arbitrary element $(x_2, y_2) \in (A \cup C) \times B$. This means there exists some $x_2 \in A \cup C$ and some $y_2 \in C$. Since $x_2 \in A \cup C$, this means that $x_2 \in A$ or $x_2 \in C$.

Case 1: In the case where $x_2 \in A$, we can say that $(x_2, y_2) \in A \times B$ because we already know $y_2 \in B$. Since $(x_2, y_2) \in A \times B$, we know that $(x_2, u_2) \in (A \times B) \cup (C \times B)$.

Case 2: Similarly, in the case where $x_2 \in C$, we can say that $(x_2, y_2) \in C \times B$ because $y_2 \in B$. Since $(x_2, y_2) \in C \times B$, we can be sure $(x_2, y_2) \in (A \times B) \cup (C \times B)$.

In both cases we have shown that $(A \cup C) \times B \subseteq (A \times B) \cup (C \times B)$ by showing that any arbitrary element in $(A \cup C) \times B$ is also in $(A \times B) \cup (C \times B)$ as desired.

Since we have shown that both $(A \times B) \cup (C \times B) \subseteq (A \cup C) \times B$ and $(A \cup C) \times B \subseteq (A \times B) \cup (C \times B)$, we can conclude that $(A \times B) \cup (C \times B) = (A \cup C) \times B$ for all nonempty sets A, B, and C from some universal set U. ■

Problem 6

> Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f$ is a surjection, then g is a surjection.

Proof. Let's assume A, B, and C are nonempty sets. We will also assume that $g \circ f$ is a surjection and will show that g is a surjection.

To prove that g is a surjection, we must show every element in its codomain has a corresponding pre-image in the domain. We will begin by choosing an arbitrary $y \in C$ (the codomain of g).

Now that we know there exists some element $y \in C$ and we assumed that $g \circ f$ is a surjection, this means that there exists an element $x \in A$ such that $g \circ f(x) = y$.

We will now construct a $b \in B$ (the domain of g) such that $g(b) = y$. Consider $b = f(x)$. Note that the function f maps elements from A to B. We know that $x \in A$, therefore the result of $f(x)$ will be an element in B. This $b \in B$. Now let's plug this value into the function g to get

$$\begin{aligned} g(b) &= g(f(x)) \\ &= g \circ f(x). \end{aligned}$$

Recall that $g \circ f(x) = y$. We can now substitute this into our equation to get

$$g(b) = y.$$

This means that any arbitrary element in the codomain of g has a pre-image in the domain, therefore g is a surjection, as desired.

Thus we have shown that for nonempty sets A, B, and C and functions $f : A \to B$ and $g : B \to C$; if $g \circ f$ is a surjection, then g is a surjection. ■

> Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be functions. If g is a surjection, then $g \circ f$ is a surjection.

Statement. We can provide a counter example to verify this statement is false. This means that we can provide a specific example where g is a surjection but $g \circ f$ is not a surjection. Consider the piecewise functions represented in Figure 3. That is, let $A = \{a\}$, $B = \{1, 2\}$ and $C = \{X, Y\}$. Functions f and g are defined such that $f(a) = 1$, $g(1) = X$, and $g(2) = Y$.

Note that g is a surjection because every element in it's codomain, C, has a corresponding element in the domain, B. Specifically, we can say $g(1) = X$ and $g(2) = Y$; thus our hypothesis is true.

However, note that $g \circ f$ is not a surjection. We say this because there is an element in the codomain of $g \circ f$, C, that has no corresponding pre-image in the domain, A. As you can tell from Figure 3, $Y \in C$ but no element from A will yield Y as an output.

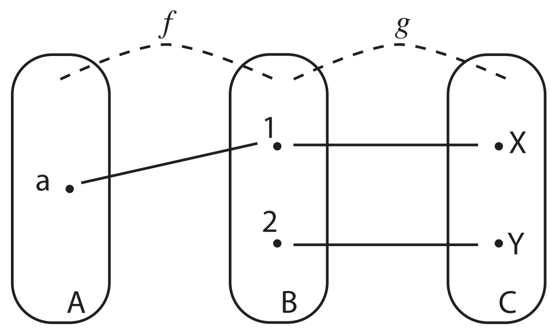

Figure 3: $f : A \to B$ and $g : B \to C$ where g is surjective

Since we have shown a specific case where the hypothesis is true, but the conclusion is false; the statement must be false.

> Let A, B, and C be nonempty sets and let $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f$ is a surjection, then f is a surjection.

Statement. We can provide a counterexample to verify this statement is false. That is, we can provide a specific example where $g \circ f$ is a surjection but f is not a surjection. Consider the piecewise functions represented in Figure 4. That is, let $A = \{a\}$, $B = \{1, 2\}$ and $C = \{X\}$. Functions f and g are defined such that $f(a) = 1$, $g(1) = X$, and $g(2) = X$.

Note that $g \circ f$ is a surjection because even element in its codomain, C, has a corresponding pre-image in the domain, A. This is easy to show because there is only one element in C, X, and it's pre-image is a which is in A.

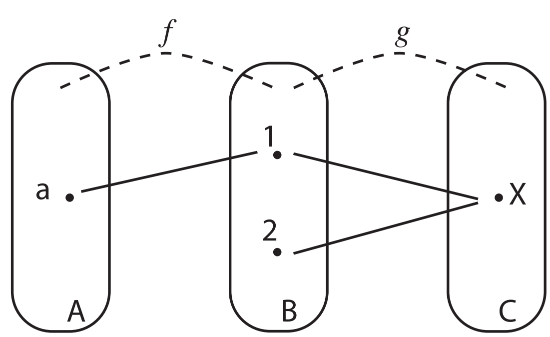

Figure 4: $f : A \to B$ and $g : B \to C$ where $g \circ f(x)$ is surjective

However, note that f is a not surjection because we can find a specific element that is in the codomain that does not have a pre-image in the domain. In this case, $2 \in B$ but there is no element in A that would produce a 2 when plugged into f.

Since we have shown a specific case where the hypothesis is true, but the conclusion is false; the statement must be false.

Problem 7

Let (f_n) be the Fibonacci sequence. For all f_m, where m is a natural number such that $m \not\equiv 0 \pmod 3$, f_m is odd.

Proof. We will assume that (f_n) is the Fibonacci sequence. The first two terms of the Fibonacci sequence, f_1 and f_2, are defined as 1. Any subsequent term in the sequence can be found by adding the two previous terms, or, symbolically, $f_n = f_{n-1} + f_{n-2}$ for all integers n greater than or equal to 3. We will assume that f_m is a number in the Fibonacci sequence where m is a natural number such that $m \not\equiv 0 \pmod 3$. This means that either $m \equiv 1 \pmod 3$ or $m \equiv 2 \pmod 3$. We will show that f_m is odd. We will prove each of these cases through mathematical induction.

Case 1: First we will examine f_m where $m \equiv 1 \pmod 3$. Using the definition of congruence and divides, we know

$$\begin{array}{rcl} m & \equiv & 1 \pmod 3 \\ 3 & \mid & m-1 \\ 3j & = & m-1 \end{array}$$

for some integer j. Solving for m gives us

$$3j+1 = m.$$

This means we are looking looking at numbers in the sequence in the form of f_{3j+1}. By definition, $3j+1$ must be a natural number which means that we must consider all cases where $j \geq 0$. To proceed with our proof by induction, we will define our predicate $P(k)$ to be

$$f_{3k+1} \text{ is odd.}$$

for all integers $k \geq 0$. We will show that $P(0)$ is true and will show that if $P(k)$ is true, then $P(k+1)$ is true.

First, we must show $P(0)$ is true. Note that if we plug 0 in for k into $3k+1$, we get 1. Therefore we are looking at the term f_1 in the sequence. By the definition of the Fibonacci sequence, the term f_1 is 1, and 1 is odd. Thereby $P(0)$ is true, as desired.

Next, we will assume that $P(k)$ is true and will show that $P(k+1)$ is true. We start by choosing an arbitrary $k \in \mathbb{Z}$ where $k \geq 0$. Assuming that $P(k)$ is true means that f_{3k+1} is odd. Thus, this can be written as

$$f_{3k+1} = 2g+1$$

for some integer g. Now let's look at $P(k+1)$, which deals with $f_{3(k+1)+1}$, or simply f_{3k+4}. The definition of the Fibonacci sequence tells us that

$$f_{3k+4} = f_{3k+3} + f_{3k+2}.$$

for all terms where $3k+4 \geq 3$. In the same way, we can rewrite f_{3k+3} to obtain

$$\begin{array}{rcl} f_{3k+4} & = & (f_{3k+2} + f_{3k+1}) + f_{3k+2} \\ & = & 2 \cdot f_{3k+2} + f_{3k+1} \end{array}$$

We can substitute the value for f_{3k+1} we found using our inductive hypothesis into this equation, and do some rearranging to get

$$\begin{aligned} f_{3k+4} &= 2 \cdot f_{3k+2} + (2g+1) \\ &= 2 \cdot f_{3k+2} + 2g + 1 \\ &= 2(f_{3k+2} + g) + 1. \end{aligned}$$

Note that f_{3k+2} is an integer because all numbers in the Fibonacci sequence are integers. We already know g in an integer; therefore $f_{3k+2} + g$ is an integer. This means that f_{3k+4} is odd by the definition of odd, as desired.

Thus, by the Extended First Principle of Mathematical Induction, we have prove that for all f_m where m is a natural number such that $m \equiv 1 \pmod 3$, f_m is odd.

Case 2: Next we will examine f_m where $m \equiv 2 \pmod 3$. Using the definition of congruence and divides, we know

$$\begin{aligned} m &\equiv 2 \pmod 3 \\ 3 &\mid m - 2 \\ 3q &= m - 2 \end{aligned}$$

for some integer q. Solving for m gives us

$$3q + 2 = m.$$

This means we are looking looking at numbers in the sequence in the form of f_{3q+2}. By definition, $3q + 2$ must be a natural number which means that we must consider all cases where $q \geq 0$. To proceed with our proof by induction, we will define our predicate $P(k)$ to be

$$f_{3k+2} \text{ is odd.}$$

for all integers $k \geq 0$. We will show that $P(0)$ is true and will show that if $P(k)$ is true, then $P(k+1)$ is true.

First, we must show $P(0)$ is true. Note that if we plug 0 in for k into $3k + 2$, we get 2. Therefore we are looking at the term f_2 in the sequence. The term f_2 is defined by the Fibonacci sequence to be 1. We know that 1 is odd, therefore $P(0)$ is true, as desired.

Next, we will assume that $P(k)$ is true and will show that $P(k+1)$ is true. We start by choosing an arbitrary $k \in \mathbb{Z}$ where $k \geq 0$. Assuming that $P(k)$ is true means that f_{3k+2} is odd. Thus, this can be written as

$$f_{3k+2} = 2p + 1$$

for some integer p. Now let's look at $P(k+1)$, which deals with $f_{3(k+1)+2}$, or simply f_{3k+5}. The definition of the Fibonacci sequence tells us that

$$f_{3k+5} = f_{3k+4} + f_{3k+3}.$$

for all terms where $3k+5 \geq 3$. In the same way, we can rewrite the terms for f_{3k+3} to obtain

$$\begin{aligned} f_{3k+5} &= (f_{3k+3} + f_{3k+2}) + f_{3k+3} \\ &= 2 \cdot f_{3k+3} + f_{3k+2}. \end{aligned}$$

We can substitute the value for f_{3k+2} we found using our inductive hypothesis into this equation, and do some rearranging to get

$$\begin{aligned} f_{3k+5} &= 2 \cdot f_{3k+3} + (2p+1) \\ &= 2 \cdot f_{3k+3} + 2p + 1 \\ &= 2(f_{3k+3} + p) + 1. \end{aligned}$$

Note that f_{3k+3} is an integer because all numbers in the Fibonacci sequence are integers. We already know p in an integer therefore $f_{3k+3} + p$ is an integer. This means that f_{3k+5} is odd by the definition of odd, as desired.

Thus, the the Extended First Principle of Mathematical Induction, have prove that for all f_m where m is a natural number such that $m \equiv 2 \pmod 3$, f_m is odd.

We have proven that for all terms f_m in the Fibonacci sequence where m is a natural number such that $m \equiv 1 \pmod 3$ or $m \equiv 2 \pmod 3$, or equivalently, $m \not\equiv 0 \pmod 3$, f_m is odd. ■