Prove $\lim_{x \to a} x^2 = a^2$.

Proof. We will show that for all $\epsilon > 0$, there exists some positive real δ such that if $|x - a| < \delta$, then $|x^2 - a^2| < \epsilon$. Let us consider the case where $|x - a| < 1$. This implies that

$$\begin{gathered} -1 < x - a < 1 \\ a - 1 < x < a + 1 \\ 2a - 1 < x + a < 2a + 1 \end{gathered}$$

Let $M = \max\{|2a + 1|, |2a - 1|\}$. This means that $|x + a| < M$ when $|x - a| < 1$.

Let $\delta = \min\left\{\frac{\epsilon}{M}, 1\right\}$. We will now show that if $|x - a| < \delta$, then $|x^2 - a^2| < \epsilon$. Since $|x - a| < \delta$, that means that either $|x - a| < 1$ or $|x - a| < \frac{\epsilon}{M}$. Let's consider the first case. If $\delta = 1$, then $\frac{\epsilon}{M}$ must be greater than 1. Thus

$$|x - a| < 1 < \frac{\epsilon}{M}.$$

In the case where $\delta = \frac{\epsilon}{M}$. That means that 1 is larger than $\frac{\epsilon}{M}$; therefore

$$|x - a| < \frac{\epsilon}{M} < 1.$$

In both cases $|x - a|$ is less than 1 and less than $\frac{\epsilon}{M}$. Thus we can simply assume that $|x - a| < \frac{\epsilon}{M}$. Since $|x - a| < 1$, we know that $|x + a| < M$. This means that

$$\begin{gathered} |x - a||x + a| < \frac{\epsilon}{M} \cdot M \\ |(x - a)(x + a)| < \epsilon \\ |x^2 - a^2| < \epsilon. \end{gathered}$$

Thus we have shown that $\lim_{x \to a} x^2 = a^2$. ■